

Reading and comprehending mathematical arguments

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Thanks

- To Chris Rasmussen and the organizers for putting this conference together and inviting me to speak
- To the Proof Comprehension Research Group (PCRG) at Rutgers University, especially:
 - Pablo Mejia-Ramos
 - Aron Samkoff
- To the National Science Foundation for funding this research:
 - (CAREER grant: DRL0643734)

Slides and cited papers for this talk
are available at:

<http://pcrg.gse.rutgers.edu>

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Proof: Let n be an integer.

$$n^3 - n = (n - 1) \cdot n \cdot (n + 1)$$

Since $n-1$, n , and $n+1$ are three consecutive integers, one of these integers is even and one of these integers is divisible by 3.

Thus, 2 divides $n^3 - n$ and 3 divides $n^3 - n$.

Hence, $n^3 - n$ is divisible by 6. \square

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Claim: $n^3 + 3n^2 + 2n$ is divisible by 6 for all natural numbers n .

Proof: Let n be an integer.

$$n^3 + 3n^2 + 2n = n \cdot (n + 1) \cdot (n + 2)$$

Since n , $n+1$, and $n+2$ are three consecutive integers, one of these integers is even and one of these integers is divisible by 3.

Thus, 2 divides $n^3 + 3n^2 + 2n$ and 3 divides $n^3 + 3n^2 + 2n$.

Hence, $n^3 + 3n^2 + 2n$ is divisible by 6. \square

Claim: $n^3 - 4n$ is divisible by 48 for all even numbers n .

Proof: Let n be an even integer.

$$n^3 - 4n = (n - 2) \cdot n \cdot (n + 2).$$

Since $n-2$, n , and $n+2$ are three consecutive even integers, one factor of n^3-4n is a multiple of 3 and one is a multiple of 4.

Since n^3-4n is the product of three even numbers and one multiple of 4, $16|n^3-4n$.

Hence, $n^3 - n$ is divisible by 48. \square

Claim: $n^3 - 4n$ is divisible by 4 for all natural numbers n .

Proof: Let n be an integer.

$$n^3 - n = (n - 1) \cdot n \cdot (n + 1).$$

Because any j consecutive integers contains a multiple of j ,

Since $n-1$, n , and $n+1$ are three consecutive integers, one of these integers is even and one of these integers is divisible by 3.

Because if d divides a factor of m , then d divides m

Thus, 2 divides $n^3 - n$ and 3 divides $n^3 - n$.

Because if p and q are coprime, $p|a$, and $q|a$, then $pq|a$

Hence, $n^3 - n$ is divisible by 6. \square

Claim: $n^3 - n$ is divisible by 6 for all natural numbers n .

To show a divides a polynomial $q(n)$:

Proof: Let n be an integer. (1) Decompose $q(n)$ into factors.

$$n^3 - n = (n - 1) \cdot n \cdot (n + 1).$$

(2) Decompose a into prime factors $p_1^{b_1}, p_2^{b_2}, \dots, p_m^{b_m}$.

Since $n-1$, n , and $n+1$ are three consecutive integers, one of these integers is even and one of these integers is divisible by 3.

(3) Use modular arithmetic to show p_m must divide a factor of a .

Thus, 2 divides $n^3 - n$ and 3 divides $n^3 - n$.

(4) Conclude a divides $q(n)$.

Hence, $n^3 - n$ is divisible by 6. \square

Claim: $n^3 - n$ is divisible by 6 for all natural numbers n .

To show this is true for $n = 16$

Proof: Let n be an integer.

$$n^3 - n = (n - 1) \cdot n \cdot (n + 1).$$

$$16^3 - 16 = 15 \cdot 16 \cdot 17.$$

Since $n-1$, n , and $n+1$ are three consecutive integers, one of these integers is even and one of these integers is divisible by 3.

$$2|16 \text{ since } 16 = 2 \cdot 8. \quad 3|15 \text{ since } 15 = 3 \cdot 5.$$

Thus, 2 divides $n^3 - n$ and 3 divides $n^3 - n$.

$$16^3 - 16 = 15 \cdot 16 \cdot 17 = (3 \cdot 5) \cdot (2 \cdot 8) \cdot 17 = (2 \cdot 3) \cdot (5 \cdot 8 \cdot 17) = 6 \cdot (5 \cdot 8 \cdot 17).$$

Hence, $n^3 - n$ is divisible by 6. \square

Motivation for talk

- Much of mathematical lectures consists of proof presentation.
 - “A typical lecture in advanced mathematics ... consists entirely of definition, theorem, proof, definition theorem proof in solemn and unrelieved concatenation” (Davis & Hersh, 1981, p. 151).
 - Observations of teachers in advanced mathematics confirm reveal proof plays an important role in math lectures (Weber, 2004; Mills, 2011; Fukawa-Connelly, in press).
- A purpose of this is to provide students with understanding.
 - Interviews with mathematicians reveal proofs are presented for many reasons, such as explanation, illustrate proof techniques, and cultural reasons (Weber, in press; Yopp, 2010).

Motivation for talk

- Students do not reap these intended learning benefits.
 - “Let one of your B students explain the statement and the proof of a theorem from the book ... My students do not have the ability to read and understand. The majority seem to simply recite the same words back” (Cowen, 1990, p. 50).
 - Math majors often cannot determine if proofs are correct (Selden & Selden, 2003; Alcock & Weber, 2005; Weber, 2010; Inglis & Alcock, 2012).
- Proof understanding is often not assessed meaningfully
 - Among nine interviewed participants, 2 asked participants to recall proofs by rote, 5 asked to prove a very similar theorem, and 2 did not assess at all. Several noted these assessments were not as meaningful as they would like (Weber, in press).

Motivation for talk

- Math professors regard studying a proof as a time-consuming and complex process (e.g., Weber & Mejia-Ramos, 2011; Weber, in press).
- Yet the interviewed mathematicians often did not focus on how to read proofs in their lectures.
 - Many did not have well developed pedagogical techniques for proof reading

Motivation for talk

- Finally, for the purposes of this conference, this is *not* just math!
- Students have difficulty learning from reading texts in other (scientific) disciplines as well (Chi, Bassok, Lewis, & Reimann, 1989; Chi, de Leeuw, Chu, & Lavancher, 1994; McNamara, 2004; Shanahan, Shanahan, & Misischia, 2011; Weinberg, 1991).

Overview of this talk

- Effective reading strategies (in mathematics)
- Unproductive beliefs that students hold about proof reading
- The importance of proof comprehension assessment
- Teaching experiments to address the problem
- Discussion about pedagogy

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What makes an effective proof-reading strategy?

- Theoretical explanation for why the strategy is useful.
- The strategy should be indicative of mathematical reasoning.
 - The majority of mathematics professors who teach proof should want students to use this strategy.
- Mathematics majors should not currently be in the habit of using this strategy.
 - Less than 50% of mathematical majors claim to engage in this behavior.

Data sources for these strategies

TASK-BASED INTERVIEWS WITH STRONG STUDENTS

- Observations of two pairs of talented and thoughtful math majors reading proofs with one another.
 - These math majors participated in other math ed studies in the university.
 - After each proof, they were given a test on how well they comprehended the proof (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012). They performed nearly perfectly.
- This was used to generate strategies and provide theoretical reasons for why they might be useful.

Data sources for these strategies

SURVEY OF STUDENTS AND MATHEMATICIANS' VIEWS ON PROOF READING STRATEGIES

- After these strategies were generated, 175 math majors who completed a proof intensive course completed an on-line survey asking if they used these strategies.
- After these strategies were generated, 83 math professors who taught a proof-intensive class completed a survey on the strategies they wanted students to use.

Data sources for these strategies

FOR STUDENTS:

A. When reading a proof of a theorem, I usually try to think about how I might prove the theorem myself BEFORE reading the proof.

B. I do not usually try to prove a theorem before reading its proof. A reason for reading a proof is to see why the theorem is true.

I strongly prefer statement A, prefer statement A, am neutral between statements A and B, prefer statement B, strongly prefer statement B

Data sources for these strategies

FOR MATHEMATICIANS:

A. When reading a proof of a theorem, I would prefer if mathematics majors think about how they might prove the theorem themselves BEFORE reading the proof.

B. I would prefer that mathematics majors not try to prove a theorem before reading its proof. A reason for reading a proof is to see why the theorem is true.

I strongly prefer statement A, prefer statement A, am neutral between statements A and B, prefer statement B, strongly prefer statement B

Data sources for these strategies

- One threat to the validity of this study is that students' reflections of what they will do will not be indicative of what they do.
- “What students say about how they read proofs seems to be a poor indicator of whether they can validate proofs reliably. They tend to ‘talk a good line’ ... However their first reading judgments yield no better than chance results”. (Selden & Selden, 2003, p. 27).
- This suggests that the survey will *overestimate* the effective proof reading strategies that students use, so this will not question results about what students fail to do.

What makes an effective proof-reading strategy

- Generation of useful strategies
TASK-BASED INTERVIEWS WITH STRONG STUDENTS
- Theoretical explanation for why the strategy is useful.
TASK-BASED INTERVIEWS WITH STRONG STUDENTS
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SURVEY OF STUDENTS AND MATHEMATICIANS' VIEWS
- Mathematics majors should not currently be in the habit of using this strategy.
SURVEY OF STUDENTS AND MATHEMATICIANS' VIEWS

What makes an effective proof-reading strategy

Proof reading strategies:

- Try to prove the theorem before reading its proof
- Attend to logical structure
- Separate the proof into separate components or sub-proofs
- Consider examples for statements that are problematic
- Compare the proof of the theorem to ones own approach

Strategy #1: Trying to prove the theorem before reading its proof

- The good students would often try to prove the theorem statement before reading its proof.
 - This motivated them to read the proof
 - They appreciated why the “obvious” approach to the proof would not work
 - They skimmed the routine parts of the proof and focused on where new techniques or sophisticated ideas were used.
 - By setting up the proof, they can appreciate what the theorem was asserting, what needed to be assumed, and what needed to be proved.

Strategy #1: Trying to prove the theorem before reading its proof

- A. When reading a proof of a theorem, I usually try to think about how I might prove the theorem myself BEFORE reading the proof.
- B. I do not usually try to prove a theorem before reading its proof. A reason for reading a proof is to see why the theorem is true.

	Prefer A:	Prefer B:	Mean:
Math:	88%	04%	1.37*
UG:	30%	55%	-0.36*

$P \iff Q$

The number of divisors of a positive integer n is odd if and only if n is a perfect square.

Proof:

1. Let d be a divisor of n .
2. Then $\frac{n}{d}$ is also a divisor of n .
3. Suppose n is not a perfect square.
4. Then $\frac{n}{d} \neq d$ for all divisors d , so we can pair up all divisors by pairing d with $\frac{n}{d}$.
5. Thus, n has an even number of divisors.
6. On the other hand, suppose n is a perfect square.
7. Then $\frac{n}{d} = d$ for some divisor d .
8. In this case, when we pair divisors by pairing d with $\frac{n}{d}$, d will be left out, so n has an odd number of divisors.

P	\iff	Odd
$\neg P$		Even
Q	\iff	k^2
$\neg Q$		$\neg k^2$

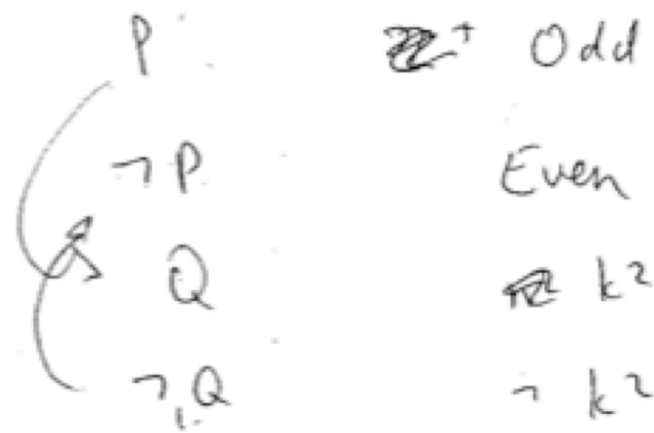


P \longleftrightarrow Q

The number of divisors of a positive integer n is odd if and only if n is a perfect square.

Proof:

1. Let d be a divisor of n .
2. Then $\frac{n}{d}$ is also a divisor of n .
3. Suppose n is not a perfect square.
- \Rightarrow 4. Then $\frac{n}{d} \neq d$ for all divisors d , so we can pair up all divisors by pairing d with $\frac{n}{d}$.
5. Thus, n has an even number of divisors.
6. On the other hand, suppose n is a perfect square.
- \Leftarrow 7. Then $\frac{n}{d} = d$ for some divisor d .
8. In this case, when we pair divisors by pairing d with $\frac{n}{d}$, d will be left out, so n has an odd number of divisors.



Strategy #2: Attending to logical structure

- A. When I read a proof, I first consider what is being assumed, what is being proven, and what proof techniques is being used.
- B. When I read a proof, I first consider how each new statement can be derived from previous statements.

	Prefer A:	Prefer B:	Mean:
Math:	64%	13%	0.81*
UG:	33%	51%	-0.25*

Strategy #3: Partitioning the proof into independent parts

- A. When I read a long proof, I try to break it into parts or sub-proofs.
- B. When I read a long proof, I do not try to break it into parts or sub-proofs but try to understand how each new line follows from previous assertions.

	Prefer A:	Prefer B:	Mean:
Math:	88%	05%	1.37*
UG:	37%	47%	-0.17

Strategy #4: Considering specific examples of confusing statements

- A. When I read a new assertion in a proof, I sometimes check if that assertion is true with a numerical example.
- B. When I read a new assertion in a proof, I check to see if it is a logical consequence of previous assertions. I do not check assertions with specific examples because you cannot prove by example.

	Prefer A:	Prefer B:	Mean:
Math:	71%	12%	0.88*
UG:	43%	34%	0.11

Strategy #5: Comparing proof technique to one's own approach

- A. When I read a proof, I compare how the methods used in the proof compares to methods I would use to prove the theorem.
- B. When I read a proof, I try not to consider how I would approach the proof, but focus on what methods was used in the actual proof.

	Prefer A:	Prefer B:	Mean:
Math:	87%	02%	1.47*
UG:	26%	50%	-0.33*

Strategies for reading explanations in the life sciences

- Identify the grain size of the phenomenon being described in a statement of the argument (e.g., gene, protein, phenotype)
- Focus on relationships between grain sizes (e.g., how do genes create traits via proteins)
- Be able to trace where matter, energy, and information went. These things do not disappear, and one should know where they went after a reaction.

Beliefs and responsibility in reading proofs

- Proving can be viewed as an interactional accomplishment in which the prover and the audience mutually engage in work to gain conviction and comprehension (e.g., Yackel & Cobb, 1994).
- Proofs cannot contain every detail. The readers need to do some work in deciphering a proof (e.g., Davis, 1972; Weber & Alcock, 2005; for science texts, see also Chi et al, 1994). The question is, how much work should the student be expected to do?
- If students and mathematicians have different views on students' responsibility, students' understanding of proofs might be limited (cf., Herbst & Brach, 2006).

Proof reading responsibility in mathematical practice

- In mathematical practice, reading a proof requires a great deal of effort
 - Some mathematicians spend over 80 hours to referee a paper, and reviews typically take 6 to 8 months to complete (Geist et al, 2011)
- However, there is also substantial variance in mathematicians' perceived sense of responsibility.
 - Some mathematicians believe it is their responsibility to check the details when refereeing, while others think it is the authors' (Geist et al, 2010).
 - Some check every line of a proof while others do not (Mejia-Ramos & Weber, submitted)

Proof reading responsibility in mathematics pedagogy

- In mathematical classrooms (as will be shown), mathematics professors believe proof comprehension is, or should be, a lengthy complicated process for students (Weber, in press).
- However, studies suggest that there is also substantial variance in mathematicians' perceived sense of responsibility (Lai, Weber, & Mejia-Ramos, 2012).
 - Mathematicians do not seem to agree on what justifications students are capable of making.
 - There is a disagreement about whether relatively easy steps in a proof should be explained to students, or whether they would benefit from constructing this understanding themselves.

Sources of evidence

PROOF VALIDATION STUDY WITH MATH MAJORS

- 28 math majors were asked to determine if 10 arguments were valid proofs and then were asked questions about their beliefs about proof reading (Weber, 2010, submitted).
 - This was used to generate hypotheses about what students believed.

INTERVIEWS WITH MATHEMATICIANS

- 9 mathematicians were interviewed about their own proof reading and their teaching with regard to proof presentation
 - This was used to generate hypotheses about differences between mathematicians and students viewpoints.

Sources of evidence

SURVEY OF STUDENTS AND MATHEMATICIANS' VIEWS ON PROOF READING STRATEGIES

- 175 math majors who completed a proof intensive course completed an on-line survey asking them questions about their beliefs on proof.
- 83 math professors who taught a proof-intensive class completed a survey on the beliefs about proof they would like their students to have.

Sources of evidence

- Generating hypotheses about differential beliefs about students' responsibility when reading proofs in class.

PROOF VALIDATION STUDY WITH MATH MAJORS

INTERVIEWS WITH MATHEMATICIANS

- Verifying that the majority of mathematicians held one view while mathematics majors held another

SURVEY OF STUDENTS AND MATHEMATICIANS' VIEWS

ON PROOF READING STRATEGIES

Belief #1: How explicit should justifications be?

- 28 math majors were asked what made a good proof:
 - 16 emphasized that a proof should have all logical details and justifications included.
 - P9: It's got to be really detailed. **You have to tell every detail**. Every step, it is very clear. I like doing things step by step.
I: So you like having every detail spelled out as much as possible?
P9: Yeah, yeah, yeah.
 - P10: Well, one it has to cover all the bases so that it is in fact a complete rigorous proof. For me, as a student, what else I would like to see is **all the intermediate sorts of steps**, things to help along, **graphs, visual things**. Things that recalled facts that perhaps I should know but you know, maybe not immediately at the tip of my tongue. That's to me what makes a good mathematical argument.

Belief #1: How explicit should justifications be?

M: I'm doing a reading course with a student on [deleted for anonymity]. So **this is one where he's deliberately not drawing pictures because he wants the reader to draw pictures.** And **so I'm constantly writing in the margin, and trying to get the student to adopt the same pattern.** Each assertion in the proof basically requires writing in the margin, or **doing an extra verification**, especially when an assertion is made that is not so obviously a direct consequence of a previous assertion.

Int: So you're writing a lot of sub-proofs?

Math: **I write lots of sub-proofs.** And also I try to check examples, especially if it's a field I'm not that familiar with, I try to check it against examples that I might know.

Belief #1: How explicit should justifications be?

- A. In a good proof, every step is spelled out for the reader. The reader should not be left wondering where the new step in the proof came from.
- B. When reading a good proof, I expect I will have to do some of the work to verify the steps in the proof myself.

	Prefer A:	Prefer B:	Mean:
Math:	27%	52%	-0.31*
UG:	75%	14%	1.06*

Belief #2: When is a proof understood?

- Once students checked every step of the proof, they rarely went back and re-read it. They did not summarize it or check the proof with examples.
- The interviewed mathematicians indicated there were two levels of understanding a proof, a logical level and in terms of its high-level ideas.

Belief #2: When is a proof understood?

If I can say how each statement in a proof logically follows from previous statements, then I understand the proof completely.

Do you agree or disagree with this statement?

	Agree:	Disagree:	Mean:
Math:	23%	67%	-0.60*
UG:	75%	13%	0.88*

Belief #3: Time spent reading a proof

How long should you typically spend studying a proof that is presented to you in your classes?

Math average (N=52): 30* to 38* minutes

Student average (N=156): 17 to 20 minutes

*- t-tests reveal min and max times significantly different between the two groups

Belief #4: Should all diagrams be included?

- A. When reading a good proof, if a diagram can help my understanding, it should be included. I should not be expected to draw a diagram myself.
- B. When reading a good proof, sometimes diagrams are not included. I expect to have to sometimes have to draw these diagrams myself.

	Prefer A:	Prefer B:	Mean:
Math:	32%	50%	-0.20
UG:	66%	19%	0.78*

Other beliefs

I do not see much value on proofs that do not help me on my homework assignments.

Do you agree or disagree with this statement?

	Agree:	Disagree:	Mean:
Math:	06%	92%	-1.57*
UG:	25%	58%	-0.55*

Other beliefs

- A. I think the most important reason for reading a proof is so I can believe that a statement is true.
- B. I think there are more important reasons for reading a proof than seeing if a statement is true.

	Prefer A:	Prefer B:	Mean:
Math:	06%	86%	-1.40*
UG:	22%	60%	-0.64*

Possible unproductive beliefs in the life sciences

- Phenomenon at one grain size (e.g., proteins) will not have effects at other levels of granularity (e.g., the environment).
- Random effects at one level of granularity (e.g., neuron firings) cannot have systematic effects at higher levels of granularity (e.g., mental illness).
- Mathematical interpretations are not necessary to comprehend texts in the life sciences. (Or diagrams and figures can be ignored).

Assessment

- Students' understanding of proofs should be assessed:
 - “You get what you assess. You do not get what you do not assess” (Resnick & Resnick, 1992, p. 59)
 - This emphasizes to students that proof reading is important.
 - This will inform students about how they should read a proof. The questions can lead students not to focus on local logical aspects of the proof, but also to global factors as well.
 - This can provide teachers with crucial information, such as how effective were their lectures and what particular issues do students find difficult.

Assessment

- Our assessment model (Mejia-Ramos, Fuller, Weber et al, 2012)
 - Local aspects:
 - The meaning of the theorem and statements in the proof
 - How is each statement justified
 - What proof technique (e.g., contraposition) is being used and how does this relate to the assumptions and conclusions
 - Global aspects
 - Summarizing the proof
 - Breaking the proof up into independent modules or sub-proofs
 - Applying the ideas of the proof to specific examples
 - Identifying and transferring the methods of the proof

Teaching experiments

The truth about our teaching experiments

- Students found the strategies valuable and we have great clips of these strategies leading to genuine insights.

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The whole truth about our teaching experiments

- Some students will not use the strategies without prodding from us, their implementation is problematic, and they don't learn that much.

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What we've learned

- We need to be more prescriptive about some of our questions. It's not enough to name a proving approach prior to reading a proof, but also to identify areas of difficulty.
- Students need immediate feedback on their strategy implementation, but providing it leads to an IRE method of instruction, causing students to disengage. Our solution, have students reach a consensus within their group before presenting their work to the professor.

Discussion

- Mathematics educators speak of a “hidden curriculum” where students “learn” undesirable beliefs based on patterns in their teaching.
 - When we ask elementary school students to do 20 math problems for homework a night, they “learn” that math problems should be solved within five minutes (Schoenfeld, 1985).
- Classroom presentations of class may encourage the beliefs
 - Students are rarely assessed, signifying a lack of importance
 - Proofs are usually presented quickly, signifying they can be studied quickly as well.
 - The emphasis on proofs is typically on correctness, suggesting other factors (summaries, partitioning) might be less important.

Discussion: Beliefs

- Proofs in high school geometry are usually presented in a two column format, with the left side containing statements, and the right side containing rules of inference stating how they were derived.
 - This format leads students to value form over substance (Schoenfeld, 1988)
 - It requires all inferences be justified.
- There seems to be an epistemological difference between the proofs we present to students and the proofs they hand in.
 - The proofs we present for students are for the purpose of communication, meant to provide insight.
 - The proofs students hand in are done for credit, with the purpose of being complete and showing they understand the nature of proof (Herbst & Brach, 2006).

Thanks